A General Method of Obtaining the Finite Integral of any Rational Algebraic Function of x; or Summing any Series of which such a Function is the General Term. By WILLIAM ORCHARD, ESQ., *Fellow of the Institute of Actuaries.*

LET $u_x=a_nx^n+a_{n-1}x^{n-1}+\ldots\ldots+a_1x+a_0$ be the given function, then, by a well known theorem,

$$u_x=u_0+x\Delta u_0+\frac{x(x-1)}{1\,.\,2}\Delta^2u_0+\frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}\Delta^3u_0+\&c., \qquad (A)$$

the integral of which, since u_0, Δu_0, Δ^2u_0, &c., are constants, and

$$\sum\frac{x(x-1)(x-2)\ldots(x-p)}{1\,.\,2\,.\,3\ldots(p+1)}=\frac{x(x-1)\ldots x-(p+1)}{1\,.\,2\,.\,3\,.\,(p+2)},$$

will be

$$\sum u_x=xu_0+\frac{x(x-1)}{1\,.\,2}\Delta u_0+\frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}\Delta^2u_0+\&c.; \qquad (B)$$

which differs from (A) only by the coefficients of Δu_0, Δ^2u_0, &c., being shifted each one place to the left.

The labour of integration is then reduced to finding the constants u_0, Δu_0, Δ^2u_0, &c., which requires the computation of u_x for $x=0=1=2=\ldots.=(n-1)=n$; these values differenced, as in the annexed form, will then determine such constants:

$$\begin{array}{llll}
u_0 & & & \\
 & \Delta u_0 & & \\
u_1 & & \Delta^2u_0 & \\
 & \Delta u_1 & & \Delta^3u_0 \\
u_2 & & \Delta^2u_1 & \\
 & \Delta u_2 & & \\
u_x & & &
\end{array}$$

The computation of u_0, u_1, u_2, &c., will be very easy, as $u_0=a_0$, $u_1=$ the algebraic sum of the coefficients, and the others requiring but trifling operations, as will appear by the examples, in which they are fully given; it will only be necessary to compute u_0, $u_1,\ldots u_{n-1}$, since from these Δu_0, $\Delta^2u_0,\ldots\Delta^{n-1}u_0$, may be found, Δ^nu_0 being known to be $=a_n\times n(n-1)\ldots2\,.\,1$.

If the coefficients of u_0, Δu_0, &c., in (B) be denoted by X_1, X_2, &c. (they are the successive binominal coefficients for index x),

$$\sum u_x=u_0\,.\,X_1+\Delta u_0\,.\,X_2+\Delta^2u_0\,.\,X^3+\&c. \qquad (C)$$

Example 1. $u_x=x^2$.

$$\begin{array}{ll}0 & \\ & 1\\ 1 & \end{array}\qquad \therefore\sum x^2=0+\frac{x(x-1)}{1\,.\,2}+2\frac{x(x-1)(x-2)}{1\,.\,2\,.\,.\,3}.$$

This particular case may be more compactly represented thus—

$$\sum x^2=\frac{x(x-1)}{1\,.\,2}\left(1+\frac{2(x-2)}{3}\right)=\frac{x(x-1)(2x-1)}{1\,.\,2\,.\,.\,3}.$$

Example 2. $u_2=x^3$.

$$\begin{array}{lll} 0 & & \\ & 1 & \\ 1 & & 6 \\ & 7 & \\ 8 & & \end{array} \qquad \sum x^3=\frac{x(x-1)}{1\,.\,2}+6\frac{x(x-1)(x-2)}{1\,.\,2\,.\,.\,3}+6\frac{x(x-1)(x-2)(x-3)}{1\,.\,2\,.\,.\,3\,.\,.\,4}.$$

Example 3. $u_x=x^2+5x-4$.

$$\begin{array}{ll} -4 & \\ & +6 \\ +2 & \end{array} \qquad \sum u_x=-4x+6\frac{x(x-1)}{1\,.\,2}+2\frac{x(x-1)(x-2)}{1\,.\,2\,.\,.\,3}.$$

Example 4. $u_x=4x^3-5x^2-14x+56$
$\qquad\qquad +3 \; - \; 8 \; +40=u_2.$

$$\begin{array}{lll} +56 & & \\ & -15 & \\ +41 & & +14 \\ & -\;1 & \\ +40 & & \end{array} \qquad \sum u_x=56X_1+15X_2+14X_3+24X_4.$$

Example 5. $u_x=x^4+15x^3-\;4x^2+\;3x+\;5$
$\qquad\qquad +17 \;\; +30 \;\; +\;63+131=u_2$
$\qquad\qquad +18 \;\; +50 \;\; +153+464=u_3.$

$$\begin{array}{llll} +5 & & & \\ & +15 & & \\ +20 & & +96 & \\ & +111 & & +126 \\ +131 & & +222 & \\ & +333 & & \\ +464 & & & \end{array} \qquad \sum u_x=5X_1+15X_2+96X_3+126X_4+24X_5.$$

Example 6. $u_x=x^5+10x^4-\;7x^3+\;14x^2+\;9x-\;24$
$\qquad\qquad +12 \;\; +17 \;\; +\;48 \;\; +105 \;+\; 186=u_2$
$\qquad\qquad +13 \;\; +32 \;\; +110 \;\; +339 \;+\; 993=u_3$
$\qquad\qquad +14 \;\; +49 \;\; +210 \;\; +849 \;+3372=u_4.$

$$\begin{array}{lllll} -\;24 & & & & \\ & +\;27 & & & \\ & & +\;156 & & \\ +\;\;3 & & & +468 & \\ & +\;183 & & & +480 \\ & & +\;624 & & \\ +\;186 & & & +948 & \\ & +\;807 & & & \\ +\;993 & & +1572 & & \\ & +2379 & & & \\ +3372 & & & & \end{array}$$

$$\sum u^x=-24X_1+27X_2+156X_3+468X_4+480X_5+120X_6.$$

The value of the algebraic expressions in the preceding examples for $x=2=3$, &c., has been found by Horner's method, which is this; write down the coefficients of the expression in their usual order, with a cypher as the coefficient of any missing term, multiply the highest coefficient by the value of $x=h$, adding the product to the second, and so on, to the end;

thus, if ah^3+bh^2+ch+d were required, the successive results found would be $ah+b$, ah^2+bh+c, ah^3+bh^2+ch+d.

This process repeated, stopping each time at one term short of the preceding, until all are exhausted, gives the result of writing $x+h$ for x in any algebraic function, which has been applied by Horner to the resolution of numerical equations.

In summing a series of which u_x is the general term, $\sum u_{x+1}$ is required, which corresponds with increasing x by unit in X_1, X_2, &c.

There are many occasions on which it is necessary to obtain Δu_x, $\Delta^2 u_x$, $\Delta^3 u_x$, &c., u_x being an algebraic function.

When the degree of the expression is not higher than the third, perhaps the easiest way is to use the particular case of Taylor's Theorem $(h=1)$.

$$\mathrm{F}(x+1)-\mathrm{F}x=\Delta\mathrm{F}x=\mathrm{F}'x+\frac{\mathrm{F}x''}{1.2}+\frac{\mathrm{F}x'''}{1.2.3}+\text{\&c.};$$

but in more complex expressions it will be better to apply Horner's method, of which it is the most simple case: subtracting from the coefficients at the close of each operation, the coefficients as they stood at the commencement. An example will make this clear.

$$\begin{array}{rrrrrrl}
4x^5+ & 5x^4- & 7x^3+ & 1x^2+ & 15x- & 14 & =u_x \\
 & +9 & +2 & +3 & +18 & +4 & \\
 & +13 & +15 & +18 & +36 & & \\
 & +17 & +32 & +50 & & & \\
 & +21 & +53 & & & & \\
 & +25 & & & & & \\
\hline
 & +20x^4+ & 60x^3+ & 49x^2+ & 21x+ & 18 & =(u_{x+1}-u_x)=\Delta u_x \\
 & & +80 & +129 & +150 & +168 & \\
 & & +100 & +229 & +279 & & \\
 & & +120 & +349 & & & \\
 & & +140 & & & & \\
\hline
 & & +80x^3+ & 300x^2+ & 358x+ & 150 & =(\Delta u_{x+1}-\Delta u_x)=\Delta^2 u_x \\
 & & & +380 & +738 & +888 & \\
 & & & +460 & +1198 & & \\
 & & & +540 & & & \\
\hline
 & & & +240x^2+ & 840x+ & 738 & =\Delta^3 u_x \\
 & & & & +1080 & +1818 & \\
 & & & & +1320 & & \\
\hline
 & & & & +480x+ & 1080 & =\Delta^4 u_x \\
 & & & & & +480 & =\Delta^5 u_x.
\end{array}$$

[The Calculus of Finite Differences is coming daily into so much greater requisition in investigations of the Theory of Life Assurance, that we willingly insert Mr. Orchard's Paper, although it has no other immediate reference to the subjects usually treated of in this *Magazine*.—Ed. *A. M.*]